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A statement about the sharpness of Theorem 2 (and thus of the conjecture) can be proved using random colorings of a complete graph. The probabalistic techniques of Erdős [2] yield the following result.

Theorem 4: [1]. *For fixed $\epsilon > 0$ and positive integer t, there exists an $n_0 = n(\epsilon, t)$ and a 2-coloring of the edges of a K_n for $n \geq n_0$ such that each set of t elements fails to monochromatically dominate at least $((1/2)^t - \epsilon)n$ of the vertices of K_n.*

It would be natural to ask whether there is an analogous result when the edges of the complete graph are colored with more than 2 colors. The following example of a 3-colored complete graph shows that this is not true. For n divisible by 3 partition the vertices of a K_n into three sets A_0, A_1, A_2, each with $n/3$ vertices. For each i $(0 \leq i \leq 2)$ color all of the edges in A_i and between A_i and A_{i+1} with color i (the indices taken mod 3). In each of the colors there are $n/3$ vertices that are isolated. To monochromatically dominate all of the vertices of K_n at least $n/3 + 1$ vertices would be needed.

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PERSONAL NOTE: I am most fortunate to have had Paul Erdős as a mentor and collaborator for twenty-five years. He is, of course, internationally known for his brilliance, extraordinary productivity, unusual life style, and his eccentricities. However, I knew him as a kind and caring human being who took a personal interest in my career, and my family, colleagues, and I will miss his friendship very much.

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Torricelli's Law—an Ideal Example of an Elementary ODE

R. D. Driver

Torricelli's law for a leaking water container is an excellent example for an elementary course on ordinary differential equations. It lends itself to a discussion of uniqueness and a discussion of mathematical modeling. It also suggests a simple classroom demonstration.

1. UNIQUENESS AND NONUNIQUENESS. The differential equation $dy/dt = \sqrt{u(y)y}$ with $y(t_0) = 0$, where u is the unit step function, has infinitely many solutions for $t \geq t_0$. In an elementary course on differential equations it is tempting to dismiss this and similar equations, telling students that such nonuniqueness will arise only in artificial examples.

But this is not quite true. Evangelista Torricelli (1608–1647) found that water leaks out a small hole in the bottom of a container at a rate proportional to the square root of the depth of the water. So if the container is an upright cylinder with a small leak at the bottom and y is the depth of the water at time t, then

$$\frac{dy}{dt} = -k\sqrt{u(y)y} \tag{1}$$

for some constant $k > 0$. Except for the minus sign, this looks like the model for nonuniqueness. In fact, it exhibits nonuniqueness when solved "backwards" from $y(t_0) = 0$.

If $y(0) = y_0 > 0$ then, by separation of variables, the unique solution of (1) is

$$y(t) = \left(\sqrt{y_0} - \frac{k}{2}t\right)^2 \quad \text{for} \quad 0 \le t \le \frac{2}{k}\sqrt{y_0}. \tag{2}$$

It follows that $y(t_0) = 0$ where $t_0 \equiv (2/k)\sqrt{y_0}$. Since $dy/dt \le 0$, the solution for $t \ge t_0$ has to be $y(t) = 0$.

But if we knew only that $y(t_0) = 0$, then (1) would have many solutions. One would be $y(t) \equiv 0$. And for each $t_1 \le t_0$ another solution would be

$$y(t) = \begin{cases} \left(\frac{k}{2}t_1 - \frac{k}{2}t\right)^2 & \text{for } t < t_1 \\ 0 & \text{for } t \ge t_1. \end{cases}$$

All of these solutions make sense physically. Each represents a different way in which the container could become empty and remain empty for $t \ge t_0$.

2. AN ILLUSTRATION OF MODELING. Some textbooks explain Torricelli's law by pointing out that a drop of water falling freely from height $y > 0$ attains speed $\sqrt{2gy}$ at height 0. But the water making its way to the bottom of the container is hardly "falling freely."

For a more believable argument simply equate the potential energy lost as a small mass m of water is removed at height $y > 0$ with the kinetic energy gained as an equal mass leaves the bottom of the container with velocity v—more or less what Torricelli did. The equation $mgy = (1/2)mv^2$ yields $v = \sqrt{2gy}$. Now let A be the cross-sectional area of the cylindrical container, let a be the area of the hole at the bottom and let Δy be the change in depth during a small time increment Δt. Then $A\Delta y \cong -av\Delta t \cong -a\sqrt{2gy}\,\Delta t$. Thus

$$\frac{dy}{dt} = -\frac{a}{A}\sqrt{2g}\sqrt{y} \quad \text{where } y \ge 0. \tag{3}$$

Notes. (i) This argument fails if a is large, because then one cannot ignore the kinetic energy gained by water moving toward the hole inside the container. (ii) If the container has two or more holes at different levels the argument is more complicated. (iii) Before Torricelli, it was assumed that water would leak out of a container at a speed proportional to the depth of water in the container.

3. EXPERIMENTAL CONFIRMATION. Torricelli's law lends itself to an easy and convincing classroom demonstration. Measure and graph the depth of the water as a function of time in a suitable can with a small hole drilled in the bottom, e.g., a juice can or a coffee can. A transparent cylinder with a flat bottom would be ideal. Starting at $y(0) = y_0 > 0$, you should get a nice parabolic graph as described by (2). Specifically, if $y(T_1) = (1/2)y_0$, then (2) predicts that the can will be empty when $t = T_1/(1 - \sqrt{0.5}) \cong 3.4T_1$. Similarly, if $y(T_2) = (1/4)y_0$, the can should be

empty when $t = 2T_2$. In practice the container will not drain completely because of surface tension. This is especially troublesome if the hole is too small. A 5 mm hole works quite well.

Now compare (1) and (3). The value of k determined so that (2) fits the experimental data will be somewhat less than $(a/A)\sqrt{2g}$. J. C. Borda (1733–1799) concluded that this is because the stream of water leaving the hole has a smaller diameter than the hole. He suggested $k = 0.6(a/A)\sqrt{2g}$. Experimentally the correction factor may be somewhat larger than 0.6.

Finally note that if one introduces $z(t) \equiv \sqrt{y(t)}$ then, as long as $y(t) > 0$, (1) reduces to $z' = -k/2$ and the parabola in (2) is replaced by the straight line $z(t) = \sqrt{y_0} - kt/2$.

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1997 Nobel To Merton and Scholes

Several times since its inception in 1969, the *Sveriges Riksbank (Bank of Sweden) Prize in Economic Science in Memory of Alfred Nobel* has been awarded to mathematicians. L. V. Kantorovich was so honored in 1975, Gerard Debreu in 1983, and game theorists John Nash, John Harsanyi, and Reinhard Selten in 1994. Fischer Black would doubtless have shared the 1997 prize with Myron S. Scholes and Robert C. Merton, had he not died of cancer in 1995, at the age of 57. Together the three developed the famous Black-Scholes formula for pricing financial options. According to Black [**1**], "A key part of the option paper I wrote with Myron Scholes was the arbitrage argument for deriving the formula. Bob [Merton] gave us that argument. It should probably be called the Black-Merton-Scholes paper." The prize committee evidently agreed, and awarded the prize to the surviving members of the trio. From all reports, the friendship and various collaborations among the three were as cordial and lasting as any in the history of science.

The honorees were cited "for a new method to determine the value of [financial] derivatives." Put and call options—for which the formula was originally developed—are but two of the many types of "derivative security" now bought and sold on a daily basis in financial markets the world over. The Black-Scholes evaluation technique has often been employed, many times by Merton himself, to assess a bewildering variety of other derivatives. The rudiments of their admirably versatile method, along with the assumptions underlying it, and a small sample of the products to which it applies, are described in the article by James Case in this issue of the MONTHLY. An abbreviated form of Merton's arbitrage argument is in [**1**].

Bertil Naslund, the chairman of the Royal Swedish Academy of Sciences prize committee, and a professor of finance at the Stockholm School of Economics, said at a news conference that the Black-Scholes formula was—along with computers and information technology—among the three main reasons for the worldwide success of derivative security markets. Other experts hailed the ideas of Black, Merton, and Scholes as the most important to emerge from economic research during the last half century, and described them as "a classic example" of the impact of academic research on industrial practice. Merton stressed in interviews following the announcement of the award that the formula was developed as an almost purely intellectual exercise, at a time when options trading was but a drop in the bucket of financial activity. Virtually all major corporations now employ derivative securities in the never-ending quest to minimize their exposure to risk. By reducing corporate risk, such instruments are said by many—though certainly not all—to reduce the liklihood and destructive potential of recessions and/or depressions.

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